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BANACH'S PRINCIPLE IN CONE METRIC SPACES OVER BANACH ALGEBRA EQUIPPED WITH A BINARY RELATION

Satish Shukla* & Nikita Dubey
Department of Applied Mathematics, Shri Vaishnav Vidyapeeth Vishwavidyalaya, Indore (M.P.)

ABSTRACT

In this work, we introduce the notion of relation-theoretic φ-contractive mappings in cone metric spaces over Banach algebra which is equipped with a binary relation. Some fixed point results for such mappings are proved. Some examples are provided which illustrate the notions introduced and the results proved herein.

Keywords- *Fixed point; cone metric space; relation-theoretic contraction principle; Banach algebra; φ-contractive mapping*

I. INTRODUCTION

The notion of vector valued metric spaces was introduced by several authors with various names, e.g., cone metric spaces, K-metric spaces and K-normed spaces etc. (see, [2, 5, 7, 20, 22]). Huang and Zhang [5] introduced such spaces under the name of cone metric spaces. Apart from other, Huang and Zhang [5] defined the Cauchy sequences and the convergence of the sequences in these spaces by using the interior points of cone in Banach spaces. While, other used the notion of norms to define Cauchy sequences and the convergence of the sequences.Huang and Zhang [5] introduced some contractive conditions and proved some fixed point results in cone metric spaces. Although, in some recent papers it was shown that the fixed point results proved in the cone metric spaces are the consequences of their existing usual metric versions. In 2013, Liu and Xu [10] initiate the study of fixed point theorems in the cone metric spaces over Banach algebra. They used a vector contractive constant instead of scalar and defined the contractive conditions of mappings. They showed that the fixed point results for such contractive mappings cannot be derived from the existing usual metric version, and so, it is worth to study such fixed point results.

The study of fixed point theorems in metric spaces equipped with a partial order relation was initiated by Ran and Reurings[20] which was further improved by several authors, see, e.g., [1, 13, 14, 22] etc. Recently, Alam and Imdad [1] introduced the notion of relation theoretic contraction on the metric spaces equipped with an arbitrary binary operation and unified and generalized several known results of metric spaces.

Cone comparison functions were introduced to generalize the contractive conditions of mappings defined on a cone metric space (see, [3, 4]). Malhotra et al. [12] introduced the relation theoretic contraction principle in cone metric spaces over Banach algebra and unified the results of Alam and Imdad [1] and Liu and Xu [10]. In this paper, we use the concept of cone comparison functions and introduce the relation theoretic φ-contractive mappings on cone metric spaces over Banach algebra and prove some fixed point results for such mappings. These results unify the result of Malhotra et al. [12], Alam and Imdad [1] and Liu and Xu [10]. Some examplesare presented which illustrate the notions and results.

II. PRELIMINARIES

Let A be a real Banach algebra, i.e., A is a real Banach space in which an operation of multiplication is defined, subject to the following properties (see, [18]): for all $x, y, z \in A; a \in \mathbb{R}$

1. $x(yz) = (xy)z$;
2. $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;
3. $a(xy) = (ax)y = x(ay)$;
4. $\| xy \| \leq \| x \| \| y \|$.

In this paper, we shall assume that the Banach algebra A has a unit, i.e., a multiplicative identity e such that $ex = xe = x$ for all $x \in A$. An element $x \in A$ is said to be invertible if there is an inverse element $y \in A$ such that $xy = yx = e$. The inverse of $x \in X$ is denoted by x^{-1}. For more details we refer to [18].

The following proposition is well known and can be found, e.g., in [18].

Proposition1. Let$A$ be a real Banach algebra with the unit e and $x \in A$. If the spectral radius $\rho(x)$ of x is less than one, i.e.,

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|x^n\|^{\frac{1}{n}} < 1.$$

Then, $e - x$ is invertible. Actually,

$$(e-x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

A subset P of A is called a cone if:

1. P is closed and $\{\theta, e\} \subset P$, where θ is the zero vector of A;
2. $\alpha P + \beta P \subset P$ for all non-negative real numbers α, β;
3. $P^2 = PP \subset P$;
4. $P \cap (-P) = \{\theta\}$.

For a given cone $P \subset A$, we can define a partial ordering $\preccurlyeq$ in A with respect to $P$by $x \preccurlyeq y$ (or equivalently $y \succcurlyeq x$) if and only if $y - x \in P$. The notation $x \ll y$ (or equivalently $y \gg x$) will stand for $y - x \in P^\circ$, where P°denotes the interior of P.

The cone P is called normal if there exists a number $K > 0$ such that for all $x, y \in P$,
$x \preccurlyeq y$ implies$\|x\| \leq K\|y\|$.
The least positive value of K satisfying the above inequality is called the normal constant of P (see [5]). Note that, for any normal cone P we have $K \geq 1$ (see [17]). In the following we always assume that P is a cone in a real Banach algebra A with $P^\circ \neq \emptyset$ (i.e., the cone P is a solid cone) and $\preccurlyeq$ is the partial ordering with respect to P.

Proposition 2.(See [21]). Let P be a cone in a Banach algebra A, $a \in P$ and $b, c \in A$ are such that $b \preccurlyeq c$, then $ab \preccurlyeq ac$.

Lemma 3.(See [6, 15]).If A is a real Banach space with a solid cone P. Then:

(a) If $a \preccurlyeq \lambda a$ with $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.
(б) If $a, b, c \in A$ and $a \preccurlyeq b \ll c$, then $a \ll c$.
(c) If $u \in P$ and if $\theta \preccurlyeq u \ll c$ for each $\theta \ll c$, then $u = \theta$.
(d) If $\| x_n \| \to 0$ as $n \to \infty$, then for any $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that, $x_n \ll c$ for all $n < n_0$.

Remark 4.(See [21]). If $\rho(a) < 1$ then$\|a^n\| \to 0$ as $n \to \infty$.

Definition5.(See [10, 11,5]).Let X be a non-empty set. Suppose that the mapping $d: X \times X \to A$ satisfies:

1. $\theta \preccurlyeq d(x,y)$for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$.
2. $d(x,y) = d(y,x)$ for all $x, y \in X$.
3. $d(x,y) \preccurlyeq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space over the Banach algebra A. Cone metric space is called normal if P is a normal cone.

Definition 6.(See [5]). Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then:

1. The sequence $\{x_n\}$ converges to x whenever for each $c \in A$ with $\theta \ll c$, there is $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

2. The sequence $\{x_n\}$is a Cauchy sequence whenever for each $c \in A$ with $\theta \ll c$, there is $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m > n_0$.
3. $(X, d)$is a complete cone metric space if every Cauchy sequence in X is convergent in X.

A mapping $T: X \to X$ is called continuous at point $x \in X$ (see, [12]), if for every sequence$\{x_n\}$in X such that $x_n \to x$ as $n \to \infty$, we have $Tx_n \to Tx$ as $n \to \infty$. $T$is said to be continuous on X if it is continuous at every point of X.

Remark 7. The limit of a convergent sequence in a cone metric space with solid cone is unique. For instance, if a sequence $\{x_n\} \subset X$ has two distinct limits say $x, y \in X$, then for every $c \in A$ with $\theta \ll c$ there exists $n_0 \in X$ such that $d(x_n, x) \ll \frac{c}{2}$ and $d(x_n, y) \ll \frac{c}{2}$ for all $n > n_0$. Therefore by Lemma 2.3 we have

$$d(x, y) \preccurlyeq d(x, x_n) + d(x_n, y) \ll \frac{c}{2} + \frac{c}{2} = c$$

for all $n > n_0$. Therefore, $x = y$, which completes the proof.

Definition 8.(See [4]). Let P be a cone in Banach algebra A. A non-decreasing function $\varphi: P \to P$ is called a comparison function if it satisfies:

(i) $\varphi(\theta) = \theta$ and $\theta \prec \varphi(x) \prec x$ for all $x \in P \backslash \{\theta\}$.
(ii) If $x \in P^\circ$ then $x - \varphi(x) \in P^\circ$.
(iii) $\lim_{n\to\infty} \varphi^n(x) = \theta$ for all $x \in P \backslash \{\theta\}$.

Definition 9.(See [9]).Let X be a nonempty set. A subset $\mathcal{R}$ of $X \times X$ is called a binary relation on X. Notice that for each pair $x, y \in X$ one of the following conditions holds:

(i) $(x, y) \in \mathcal{R}$ which amounts to saying that "x is $\mathcal{R}$-related to y" or "x relates to y under $\mathcal{R}$." Sometimes, we write $x\mathcal{R}y$ instead of$(x, y) \in \mathcal{R}$.
(ii) $(x, y) \notin \mathcal{R}$, which means that "$x$ is not $\mathcal{R}$-related to y" or "x does not relate to y under $\mathcal{R}$".

Trivially,$X \times X$ and ϕ being subsets of $X \times X$ are binary relations on X, which arerespectively called the universal relation (or full relation) and empty relation. Another important relation of this kind is the relation

$$\Delta_x = \{(x, x): x \in X\},$$

andcalled the identity relation or the diagonal relation on X. Throughout this paper, $\mathcal{R}$ standsfor a nonempty binary relation, but for the sake of simplicity, we write only "binaryrelation"instead of "nonempty binary relation".

Definition 10. (See [1]).Let $\mathcal{R}$ be a binary relation defined on a nonempty set $X$and $x, y \in X$. We say that x and y are $\mathcal{R}$-comparative if either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$. We denote itby $[x, y] \in \mathcal{R}$.

Definition 11. (See [9]). Let X be a nonempty set and $\mathcal{R}$ a binary relation on X.

(1) The inverse or transpose or dual relation of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$, is defined by $\mathcal{R}^{-1} = \{(x, y) \in X \times X: y, x \in \mathcal{R}$.
(2) The reflexive closure of $\mathcal{R}$, denoted by $\mathcal{R}^{\#}$, is defined to be the set $\mathcal{R} \cup \Delta_X$ $(\text{i.e.}, \mathcal{R}^{\#} := \mathcal{R} \cup \Delta_X)$. Indeed $\mathcal{R}^{\#}$is the smallest reflexive relation on X containing$\mathcal{R}$.
(3) The symmetric closure of$\mathcal{R}$, denoted by $\mathcal{R}_s$, is defined to be the set $\mathcal{R} \cup \mathcal{R}^{-1}$$(\text{i.e.}, \mathcal{R}_s := \mathcal{R} \cup \mathcal{R}^{-1})$. Indeed, $\mathcal{R}_s$ is the smallest symmetric relation on X containing $\mathcal{R}$.

Proposition 12. (See [1]). For a binary relation $\mathcal{R}$ defined on a nonempty set X, we define the relation $\mathcal{R}_s$ by:

$$(x, y) \in \mathcal{R}_s \Leftrightarrow [x, y] \in \mathcal{R}.$$

Definition 13.(See [1]).Let X be a nonempty set and$\mathcal{R}$ a binary relation on X. Then:

(a) A sequence$\{x_n\}$in X is called$\mathcal{R}$-preserving if$(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}$.
(b) $X$the relation R is called T-closed if for every$(x, y) \in \mathcal{R}$, we have$(Tx, Ty) \in \mathcal{R}$.

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Proposition 14.(See [1]). Let X, T and $\mathcal{R}$ be the same as in Definition 2.13. If $\mathcal{R}$ is T-closed, then $\mathcal{R}_s$is also T-closed.

Definition 15.(See [8, 19]).Let X be a nonempty set and$\mathcal{R}$ a binary relation on X.

(a) Asubset E of X is called$\mathcal{R}$-directed if for each $x, y \in E$, there exists z ∈ X such that$(x, z) \in \mathcal{R}$ and$d\,(y, z) \in \mathcal{R}$.

(b) For $x, y \in X$, a path of length r (where$r$is a natural number) in $\mathcal{R}$ from x to y is a finite sequence $\{z_i\}_{i=0}^{k} \subset X$ satisfying the following conditions:

(i) $z_0 = x$ and $z_r = y$;

(ii) $(z_i, z_{i+1}) \in \mathcal{R}$ for each $0 \leq i \leq r-1$.

Definition 16.(See [12]). Let(X, d)be a cone metric space over Banach algebra A, P the underlying solid cone and $\mathcal{R}$ be a binary relation on X. Then, $\mathcal{R}$ is called d-self-closed if for every $\mathcal{R}$-preserving sequence$\{x_n\}$with $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$of$\{x_n\}$such that $[x_{n_k}, x] \in \mathcal{R}$ for all $k \in \mathbb{N}$.

I. Main results

We start this section by introducing some auxiliary notions.

Definition 1. Let(X, d)be a cone metric space over Banach algebra A, $\mathfrak{C}$ be the set of all cone comparison functions on P, $T: X \to X$ be a given mapping. We say that T is a φ-contractive mapping if there exists a comparison function $\varphi \in \mathfrak{C}$ such that

$$d(Tx, Ty) \preccurlyeq \varphi\big(d(x, y)\big) \text{for all } x, y \in X \tag{1}$$

Definition2. Let(X, d)be a cone metric space over Banach algebra A, $\mathfrak{C}$ be the set of all cone comparison functions on P, $T: X \to X$ be a given mappingand $\mathcal{R}$ be a binary relation on X. We say that T is a relation theoretic φ-contractive mapping if there exists a comparison function $\varphi \in \mathfrak{C}$ such that

$$d(Tx, Ty) \preccurlyeq \varphi\big(d(x, y)\big) \text{for all } x, y \in X \text{ with } (x, y) \in \mathcal{R}. \tag{2}$$

Remark 3. If we take $\mathcal{R}$ as universal relation, then a relation theoretic φ-contractive mapping reduces into a φ-contractive mapping, and so, the class of relation theoretic φ-contractive mappings is a generalization of the class of a φ-contractive mappings. The following examples show that this generalization is a proper generalization, as well as, the existence and uniqueness of fixed point of a relation theoretic φ-contractive mapping cannot be concluded even in a complete cone metric space.

Example 4. Let $X = \mathbb{R}$, $A = C^1_{\mathbb{R}}[0,1]$ be the real Banach algebra with point wise multiplication and norm defined by $\|x(t)\| = \|x(t)\|_\infty + \|x'(t)\|_\infty$ for all $x(t) \in A$.Let $P = \{x(t) \in A : x(t) \geq 0 \text{ for all } t \in [0,1]\}$be a solid cone in A. Define a mapping $d: X \times X \to P$ by $d(x, y) = \rho e^t$ for all $x, y \in X$, where $\rho = |x - y|$, then (X, d) is a complete cone metric space over Banach algebra A. Define a mapping $T: X \to X$ by $Tx = x$ for all $x \in X$. Define the relation $\mathcal{R}$ by $\mathcal{R} = \Delta_X$. Then, T is a relation theoretic φ-contractive mapping for any arbitrary $\varphi \in \mathfrak{C}$. On the other hand, T is not a φ-contractive mapping for every $\varphi \in \mathfrak{C}$.

Example 5. Let (X, d) as in the previous example and let $T: X \to X$ be an arbitrary mapping. Define the relation $\mathcal{R}$ by $\mathcal{R} = \emptyset$. Then, T is a relation theoretic φ-contractive mapping for any arbitrary $\varphi \in \mathfrak{C}$. On the other hand, it is easy to see that such mapping T need not to be a φ-contractive mapping for every $\varphi \in \mathfrak{C}$.
The following theorem isan existence result of the fixed point of a continuous φ-contractive mapping on cone metric spaces over Banach algebra.

Theorem 6. Let(X, d)be a complete cone metric space over Banach algebra A, and $T: X \to X$ be arelation theoretic φ-contractive mappingsatisfying the following conditions:

(i) $\mathcal{R}$ is T-closed;

(ii) there exists $x_0 \in X$ such that $(x_0, T^r x_0) \in \mathcal{R}$ for all $r \in \mathbb{N}$;

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(iii) T is continuous on X.

Then T has a fixed point, i.e., there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. Let $x_0 \in X$ be such that $(x_0, T^r x_0) \in \mathcal{R}$ for all $r \in \mathbb{N}$. Define a sequence $\{x_n\}$ in X by

$$x_{n+1} = Tx_n \text{ for all } n \in \mathbb{N}. \quad (3)$$

Therefore $(x_0, x_r) \in \mathcal{R}$ for all $r \in \mathbb{N}$. Therefore, by T-closedness of $\mathcal{R}$ we have $(Tx_0, Tx_r) = (x_1, x_{r+1}) \in \mathcal{R}$ for all $r \in \mathbb{N}$.

By induction we obtain

$$(x_n, x_{r+n}) \in \mathcal{R} \text{ forall } r, n \in \mathbb{N}.$$

If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x^* = x_n$ is a fixed point of T. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Since T is arelation theoretic φ-contractive mapping, using (2) and (3) and the properties of φ we obtain:

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\preccurlyeq \varphi(d(x_{n-1}, x_n)) \\ &= \varphi(d(Tx_{n-2}, Tx_{n-1})) \\ &\preccurlyeq \varphi^2(d(x_{n-2}, x_{n-1})) \\ &= \varphi^2(d(Tx_{n-3}, Tx_{n-2})) \\ &\preccurlyeq \varphi^3(d(x_{n-3}, x_{n-2})). \end{aligned}$$

Using the properties of comparison function φ, by induction we obtain that:

$$d(x_n, x_{n+1}) \preccurlyeq \varphi^n\big(d(x_0, x_1)\big) \text{ for all } n \in \mathbb{N}. \quad (4)$$

For $c \gg \theta$, we can choose $n_0 \in \mathbb{N}$ and $\delta > 0$ such that

$$c - \varphi(c) + \{u \in A: \|u\| < \delta\} \subset P^\circ, \|\varphi^n(d(x_0, x_1))\| < \delta$$

and $\varphi^n(d(x_0, x_1)) \ll c - \varphi(c)$ for all $n > n_0$.

Therefore, by Lemma 2.3, (4) and the above inequality we obtain

$$d(x_n, x_{n+1}) \ll c - \varphi(c) \preccurlyeq c \text{ forall } n > n_0. \quad (5)$$

We shall show that the sequence $\{x_n\}$ is a Cauchy sequence. Then, for $n > n_0$, using (3) and (5) and properties of φ we have

$$\begin{aligned} d(x_n, x_{n+2}) &\preccurlyeq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \\ &\ll c - \varphi(c) + d(Tx_n, Tx_{n+1}) \\ &\preccurlyeq c - \varphi(c) + \varphi(d(x_n, x_{n+1})) \\ &\preccurlyeq c - \varphi(c) + \varphi(c) \\ &= c. \end{aligned}$$

Similarly, for $n > n_0$ using (3) and (5) and properties of φ we obtain

$$\begin{aligned} d(x_n, x_{n+3}) &\preccurlyeq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+3}) \\ &\preccurlyeq c - \varphi(c) + d(Tx_n, Tx_{n+2}) \\ &\preccurlyeq c - \varphi(c) + \varphi(d(x_n, x_{n+2})) \\ &\preccurlyeq c - \varphi(c) + \varphi(c) \\ &= c. \end{aligned}$$

By induction, we obtain

$$d(x_n, x_{n+r}) \ll c \text{ for all } r \in \mathbb{N} \text{ and } n > n_0. \quad (6)$$

Thus $\{x_n\}$ is a Cauchy sequence in the cone metric space X. Since (X, d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. We shall show that x^* is a fixed point of T.

From the continuity of T, it follows that $x_{n+1} = Tx_n \to Tx^*$ as $n \to \infty$. In view of Remark 2.7 we obtain $x^* = Tx^*$, that is, x^* is a fixed point of T. ■

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In the next theorem, we replace the continuity of T by another hypothesis which does not depend on the nature of T.

Theorem 7. Let(X,d)be a complete cone metric space over Banach algebra A, and $T: X \to X$ be arelation theoretic φ-contractive mapping satisfying the following conditions:

(i) $\mathcal{R}$ is T-closed;
(ii) there exists $x_0 \in X$ such that $(x_0, T^r x_0) \in \mathcal{R}$ for all $r \in \mathbb{N}$;
(iii) $\mathcal{R}^s$is d-self-closed.

Then T has a fixed point, i.e., there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. Following the arguments similar to those in the proof of Theorem 3.6, we obtain that $\{x_n\}$ is a Cauchy sequence in X and $(x_n, x_{r+n}) \in \mathcal{R}$ forall $r, n \in \mathbb{N}$. By completeness of (X,d), there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. By hypothesis (iii) there exists a subsequence $\{x_{n_k}\}$ such that $(x_{n_k}, x^*) \in \mathcal{R}^s$ for all $k \in \mathbb{N}$.

Now using (2), (3) and hypothesis (iii), we obtain

$$\begin{aligned} d(Tx^*, x^*) &\preccurlyeq d(Tx^*, Tx_{n_k}) + d(Tx_{n_k}, x^*) \\ &\preccurlyeq \varphi\left(d(x^*, x_{n_k})\right) + d(Tx_{n_k}, x^*) \\ &\preccurlyeq \varphi\left(d(x^*, x_{n_k})\right) + d(x_{n_k+1}, x^*). \end{aligned}$$

Suppose $c \in P^\circ$ be given. Since $x_{n_k} \to x^*$ as $k \to \infty$,then there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_k}, x^*) \ll \frac{c}{2}, d(x_{n_k+1}, x^*) \ll \frac{c}{2}$forall $k > n_0$. So by the properties of the function φ we have $\varphi(d(x_n, x^*)) \preccurlyeq \varphi\left(\frac{c}{2}\right) \ll \frac{c}{2}$ for all $k > n_0$. Therefore, it follows from the above inequality that

$$d(Tx^*, x^*) \ll \frac{c}{2} + \frac{c}{2} = c \text{ for all } k > n_0.$$

Using Lemma 2.3 we obtain $d(Tx^*, x^*) = \theta$, i.e., $Tx^* = x^*$. Thus x^* is a fixed point of T. ∎

Example 8. Let $X = \mathbb{R}$, $A = \mathbb{R}^2$ with norm $\|x_1, x_2\| = |x_1| + |x_2|$ for all $(x_1, x_2) \in A$, multiplication defined by $(x_1, x_2) \cdot (y_1, y_2) = (x_1 y_1, x_1 y_2 + x_2 y_1)$, unity $e = (1,0)$ and cone $P = \{(x_1, x_2) \in A: x_1, x_2 \geq 0\}$. Define $d: X \times X \to A$ by $d(x,y) = (1,\alpha)|x - y|$ for all $x, y \in X$, where $\alpha > 0$. Then (X,d) is a complete cone metric space over Banach algebra A. Define a mapping $T: X \to X$ by

$$Tx = \begin{cases} \dfrac{x}{1+x} \text{if } \ x \in [0,1] \\ x - \sin x \text{ if } \ x \notin [0,1] \end{cases}.$$

Define a binary relation $\mathcal{R}$ on $X$by $\mathcal{R} = \{(x,y) \in X \times X: y \leq x, x, y \in [0,1]\}$.Then it is easy to see that T is a relation theoretic φ-contractive mapping with $\varphi(x_1, x_2) = \left(\frac{x_1}{1+x_1}, \frac{x_2}{1+x_2}\right)$ for all$(x_1, x_2) \in P$.

(i) Since T is non-decreasing with respect to usual order of $\mathbb{R}$, therefore, $\mathcal{R}$ is T-closed.
(ii) For each $x \in [0,1]$ we have$(x_0, T^r x_0) \in \mathcal{R}$.
(iii) Since $[0,1]$ is a closed subset of $\mathbb{R}$ (with respect to usual topology), therefore limit of any convergent sequence in $[0,1]$ is again in $[0,1]$. So, $\mathcal{R}^s$ is d-self-closed.

Thus, all the properties of Theorem 3.7 are satisfied and we can assure the existence of fixed point of$T$ in $\mathbb{R}$. Indeed, the set of fixed point of T is $\mathcal{F}(T) = \{(n\pi, n\pi): n \in \mathbb{Z}\}$.

The above example shows that previous theorems ensure only the existence of the fixed point of a relation theoretic φ-contractive mapping defined on a cone metric space over Banach algebra and the fixed point of such mappings need not be unique. To ensure the uniqueness of fixed point, we introduce the following condition:

(H): For all $x, y \in X$, there exists $z \in X$ such that$(x,z) \in \mathcal{R}^s$ and$(y,z) \in \mathcal{R}^s$.

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Theorem9. Adding condition (H) to the hypotheses of Theorem 3.6(resp. Theorem 3.7) we obtain uniqueness of the fixed point of T.

Proof. Suppose that x^* and $y^*$are two distinct fixed point of T. From (H), there exists $z \in X$ such that

$$(x^*, z) \in \mathcal{R}^s \text{ and} (y^*, z) \in \mathcal{R}^s.$$

Assume that $x^* \neq z$ and $y^* \neq z$. Since T is a relation theoretic φ-contractive mapping, using the T-closedness of $\mathcal{R}$ we obtain

$$\begin{aligned} d(x^*, T^n z) &= d(Tx^*, T^n z) \\ &= d(Tx^*, TT^{n-1}z) \\ &\preccurlyeq \varphi\big(d(x^*, T^{n-1}z)\big). \end{aligned}$$

Repetition of the above process and the properties of φ give:

$$d(x^*, T^n z) \preccurlyeq \varphi^n\big(d(x^*, z)\big) \text{for all } n \in \mathbb{N}.$$

Since $\lim_{n\to\infty} \varphi^n\big(d(x^*, z)\big) = \theta$, by Lemma 2.3, for every $c \in P^\circ$ there exists$n_0 \in \mathbb{N}$ such that $d(x^*, T^n z) \ll c$ for all $n > n_0$. Therefore, $T^n z \to x^*$ as $n \to \infty$.

With a similar process we can obtain $T^n z \to y^*$ as $n \to \infty$, and so in view of Remark 2.7 we obtain $x^* = y^*$. This contradiction proves the uniqueness of fixed point.

If $x^* = z$ or $y^* = z$, then the conclusion is trivial. ∎

The following corollary is an improved version of the main result of Liu and Xu [10].

Corollary 10. Let (X, d) be a complete cone metric space and P be a solid cone. Suppose that the mapping $T: X \to X$ satisfies the generalized Lipschitz condition

$$d(Tx, Ty) \preccurlyeq kd(x, y) \text{for all } x, y \in X,$$

where $k \in P$ with $\rho(k) < 1$. Then T has a unique fixed point in X, and for any $x \in X$, the iterative sequence $\{T^n x\}$converges to the fixed point.

Proof. Define$\boldsymbol{\mathcal{R}} = X \times X$, i.e., theuniversal relation and $\varphi(x) = kx$ for all $x \in P$, in Theorem 3.9 we obtain the required result. ∎

The following corollary is an improved cone metric version of results of Ran and Reurings [16] andNieto and Rodríguez-López [13].

Corollary 11. Let(X, d)be a complete cone metric space over Banach algebra A and $\sqsubseteq$ be a partial order on X. Let $T: X \to X$ be a mapping satisfying the following conditions:

(i) $d(Tx, Ty) \preccurlyeq \varphi(d(x, y))$ for all $x \sqsubseteq y$;
(ii) T is nondecreasing with respect to $\sqsubseteq$;
(iii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq Tx_0$;
(iv) at least one of the following conditions is satisfied:
 (A) T is continuous; or
 (B) if$\{x_n\}$is a non-decreasing sequencewith respect to $\sqsubseteq$ with $x_n \to x \in X$ as $n \to \infty$,then there exists a subsequence $\{x_{n_k}\}$of$\{x_n\}$such that $x_{n_k} \sqsubseteq x$ for all $k \in \mathbb{N}$.

Then T has a fixed point, i.e., there exists $x^* \in X$ such that $Tx^* = x^*$. In addition, if for all $x, y \in X$ there exists a $z \in X$, such that $x \sqsubseteq z$ and $y \sqsubseteq z$, then the fixed point of X is unique.

Proof. Define a binary relation $\mathcal{R}$ on X by $\mathcal{R} = \{(x, y) \in X \times X: x \sqsubseteq y\}$. Then, it is easy to see that all the conditions of Theorem 3.9 are satisfied and result follows. ∎

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